IB Linear Algebra – Example Sheet 4

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1. The square matrices A and B over the field F are congruent if $B = P^T AP$ for some invertible matrix P over F. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

   If A is the matrix of the first of these (say), find a non-singular matrix P such that $P^T AP$ is diagonal with entries ± 1.

3. (i) Show that the function $\psi(A, B) = \mathrm{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\mathrm{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\mathrm{tr}(AB^T)| \le \mathrm{tr}(AA^T)^{1/2}\mathrm{tr}(BB^T)^{1/2}$.
   (ii) Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form on $\mathrm{Mat}_n(\mathbb{R})$. Find its rank and signature.

4. A bilinear form $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called *skewsymmetric* if $\varphi(u, v) = -\varphi(v, u)$ for all u, v. If φ is non-degenerate, show that n is even, and that there is a basis with respect to which the matrix representation of φ is block-diagonal with blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. What is the maximum dimension of a subspace on which φ vanishes?

5. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space V.
   (i) Find the rank and signature of ψ in the case $V = \mathbb{C}^3$ and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

   (ii) Show in general that if $n > 2$ then $\psi(u, v) = \frac{1}{n}\sum_{k=1}^{n} \zeta^k \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.

6. Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite. *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.

7. An endomorphism α of a finite dimensional inner product space V is *positive definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
   (i) Prove that a positive definite endomorphism has a unique positive definite square root.
   (ii) Let α be an invertible endomorphism of V and α^* its adjoint. By considering $\alpha^*\alpha$, show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.

8. Let V be a finite dimensional complex inner product space, and let α be an endomorphism on V. Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^*. Deduce that there is an orthonormal basis of eigenvectors of α.

9. Let P_n be the ($(n + 1)$-dimensional) space of real polynomials of degree $\le n$. Define

$$(f, g) = \int_{-1}^{+1} f(t)g(t)\,dt\,.$$

Show that $(\, , \,)$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint. If f is an eigenvector of α of degree k, what is the corresponding eigenvalue? Why must α have precisely one monic eigenvector of degree k for each $0 \leq k \leq n$?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.
(i) For $i \neq j$, $(s_i, s_j) = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α.

What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? Explain why that is the case.

10. Let a_1, a_2, $\ldots$, a_n be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

11. Let S be an $n \times n$ real symmetric matrix with $S^k = I_n$ for some $k \geq 1$. Show that $S^2 = I_n$.

12. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix and $k > 0$ satisfies $|a_{i,j}| \leq k$ for all $1 \leq i, j \leq n$, then:

$$|\det(A)| \leq k^n n^{\frac{n}{2}}.$$